

Geometric monodromy and the hyperbolic disc*

Ivan Smith

New College, Oxford

Abstract

Symplectic four-manifolds give rise to Lefschetz fibrations, which are determined by monodromy representations of free groups in mapping class groups. We study the topology of Lefschetz fibrations by analysing the action of the monodromy on the universal cover of a smooth fibre and give a new and simple proof that Lefschetz fibrations arising from Donaldson's construction via pencils of sections never decompose as non-trivial fibre sums; in particular not all Lefschetz fibrations are fibre sums of holomorphic Lefschetz fibrations. We also show that there can never be isotopy classes of simple closed curve invariant under the monodromy and as a corollary we give a symplectic analogue of Manin's theorem, showing that Lefschetz fibrations admit at most finitely many homotopy classes of geometric section.

1 Introduction

A Lefschetz fibration of a smooth four-manifold X comprises a map $f : X \rightarrow \mathbb{S}^2$ with finitely many critical points, all in distinct fibres, at each of which f takes the form $(z_1, z_2) \mapsto z_1 z_2$ with respect to local complex co-ordinates. These co-ordinates (in both X and the base sphere) are obliged to respect global orientations. Away from the critical values of the map f we have a locally trivial fibre bundle with fibre a smooth surface of some genus g ; at the finitely many critical values the fibre has a single positive node. The vanishing cycles of a Lefschetz fibration are simple closed curves in a fixed fibre which shrink to the nodal points of the critical fibres; they are uniquely defined to isotopy given a choice of paths in \mathbb{S}^2 from the fixed base-point to the points of the critical set $\{\text{Crit}\}$. The monodromy homomorphism $\rho : \pi_1(\mathbb{S}^2 \setminus \{\text{Crit}\}) \rightarrow \Gamma_g$ takes the simple closed curve about one critical point to the positive Dehn twist about the corresponding vanishing cycle.

It follows, by an easy adaptation of an argument due to Thurston (e.g. [6], [13]), that the four-manifold X is symplectic with f giving rise to a distinguished deformation class of symplectic form, with respect to which all the fibres are symplectic submanifolds away from their singular points. A resurgence of interest in manifolds with this particular structure follows a remarkable theorem of Donaldson [2]: any symplectic four-manifold admits such a topological structure after finitely many symplectic blow-ups, and with the fibres being symplectic submanifolds. More particularly, Donaldson proves that certain "pencils" of sections of a line bundle with first Chern class represented by a large multiple of the symplectic form (perturbed and scaled to be an integral cohomology class) give rise to families of symplectic submanifolds all of which intersect transversely pairwise at finitely many base-points. The picture is familiar from pencils of

*Subject classification: Symplectic geometry. MSC: 53C15, 57R55.

divisors in Kähler surfaces, and as in that situation one obtains a Lefschetz fibration on blowing up these base-points. This material, and more, may be found in [13] or [14] for instance.

It follows that the symplectic Lefschetz fibrations provided by Donaldson's existence theorem have some additional structure: a number of sections of the map f each of which has image a symplectic sphere of square (-1) . The first result of this paper indicates the constraint that this imposes on the rest of the topology of the fibration. Recall that given two Lefschetz fibrations X, Y with fibres of the same genus, one may form their *fibre sum*; topologically we remove neighbourhoods of smooth fibres in each and glue the resulting (trivial) surface bundles over circles by any fibre-preserving diffeomorphism ϕ . Denote the result $X \#_{\phi} Y$. Symplectically we may perform the same operation (perhaps after a suitable scale change to make the distinguished smooth fibres have the same symplectic area). The fibre summation is said to be non-trivial provided that each of the original fibrations has a non-zero number of critical fibres (equivalently, provided the final four-manifold does not have Euler characteristic equal to that of either of the constituent pieces).

(1.1) THEOREM: *If $f : X \rightarrow \mathbb{S}^2$ contains a section of square (-1) then it cannot be decomposed as any non-trivial fibre sum.*

Gang Tian asked if simply connected Lefschetz fibrations are necessarily fibre sums of holomorphic fibrations. Rather special examples due to the author (presented later) showed this to be false in general. These examples relied on deep facts on symplectic manifolds arising from Seiberg-Witten theory. Independently, Stipsicz [15] proved the above theorem also using Seiberg-Witten techniques, and in particular the fact that a symplectic four-manifold with $b_+ > 1$ cannot contain a symplectic sphere of non-negative square. By contrast, the proof of (1.1) that we shall present in the third section is both self-contained and elementary. Note in any case that Donaldson's existence result for pencils and (1.1) imply a negative answer to Tian's question:

(1.2) COROLLARY: *There are infinitely many simply connected symplectic Lefschetz fibrations which are not fibre sums of holomorphic fibrations.*

Our method provides a nice characterisation of the self-intersection of a section of a Lefschetz fibration in terms of a certain rotation number at infinity, where the monodromy is lifted to the universal cover of a fibre (2.3).

Sections correspond to invariant zero-dimensional homology classes of a certain type. Here "invariant" means "invariant with respect to the monodromy diffeomorphisms which encode the topology of the fibration". One can also ask about invariant one-dimensional homology or homotopy classes. The monodromy homomorphism

$$\rho : \pi_1(\mathbb{S}^2 \setminus \{\text{Crit}\}) \rightarrow \Gamma_g$$

has image the mapping class group Γ_g of the generic smooth fibre. Composing with the natural map $\pi : \Gamma_g \rightarrow Sp_{2g}(\mathbb{Z})$ from the mapping class group to the symplectic group, we obtain the cohomological monodromy which describes the action of the monodromy maps on the symplectic vector space $(H^1(\text{Fibre}), \langle \cdot, \cdot \rangle)$, where the symplectic form is given by the cup-product. It is an easy fact that there is an identity $H^1(X) = H^1(F)^{\rho}$ between the first cohomology of the total space of the fibration and the invariant classes

$$H^1(F)^\rho = \{a \in H^1(F, \mathbb{Z}) \mid f^*(a) = a \ \forall f \in \text{Image}(\rho)\}.$$

In particular, if $b_1(X) > 0$ then $\pi \circ \rho$ has a trivial subrepresentation. Related properties of this cohomological monodromy encode some of the fundamental properties of Kähler as opposed to symplectic four-manifolds: the Hard Lefschetz theorem states that if X is Kähler then this invariant subspace is a symplectic subspace, in particular of even dimension.

The invariant cohomology is however at first glance quite insensitive - it can be defined from the data of the homology classes and not the isotopy classes of the vanishing cycles. One might expect to obtain refinements of results such as the Hard Lefschetz theorem by passing from invariant homology to invariant homotopy classes. Our second result, which first appeared in the author's D.Phil thesis [13] and answers a question due to Ludmil Katzarkov, is in that sense a disappointment:

(1.3) THEOREM: *Let $f : X \rightarrow \mathbb{S}^2$ be any symplectic Lefschetz fibration; write F for a fixed smooth fibre and ρ for the monodromy representation as above. Then there is no finite set of disjointly embedded simple closed curves $\{C_1, \dots, C_r\} \subset F$ whose union is preserved to isotopy by ρ . In particular, no free isotopy class of embedded curve is invariant under the monodromy.*

By way of consolation, we find that the vanishing cycles of any Lefschetz fibration must fill the fibre F (their complement is a bunch of discs)¹. If the vanishing cycles did not fill, then by inserting a large genus surface with two boundary components in place of an annular neighbourhood of a curve disjoint from the vanishing cycles, one could build a Lefschetz fibration with arbitrarily large fibre genus and fixed number of critical fibres. Thus one can deduce the “one curve” case of (1.3) from any genus-dependent bound on the minimal length of positive relations in mapping class groups; however, such bounds have been established only using gauge theory.

Two well-known finiteness theorems in the geometry of holomorphic fibrations are the Arakelov-Parsin theorem and Manin's theorem (for a treatment from our perspective see for instance [7]). The first has the particular consequence that there are only finitely many holomorphic Lefschetz fibrations with fixed fibre genus and fixed set of critical values; the second says that any non-trivial holomorphic fibration admits at most finitely many holomorphic sections. The first of these results is false in the symplectic setting; there are infinitely many homeomorphism types of symplectic manifold which admit Lefschetz fibrations by fixed genus curves with a fixed number of critical values (which can be placed at any given set of points in the sphere). Such examples are given in [13] and this phenomenon is discussed in more detail in [4]. An analogue of Manin's theorem, however, can be deduced from (1.1).

(1.4) COROLLARY: *Let $f : X \rightarrow \mathbb{S}^2$ be a Lefschetz fibration. Then only finitely many homotopy classes in X contain smooth sections of f .*

Note that this is false even for holomorphic fibre bundles over bases of higher genus: for instance, the trivial fibration $\mathbb{T}^4 \rightarrow \mathbb{T}^2$ admits infinitely many distinct homotopy classes of smooth section. (It is also false in general at the level of homology; there may be infinitely many homology classes with algebraic

¹Apparently this result was classically known to Thurston and others, but I have been unable to trace a written account.

intersection number one with the fibre.) Similarly, one can contrast with the infinite families of symplectic representatives for certain reducible homology classes - multiples of a fibre of a Lefschetz fibration - given in [12]. Any section of a Lefschetz fibration can be made a symplectic submanifold if we perturb the symplectic form by a deformation equivalence, and our proof shall imply that the symplectic isotopy class of a section is determined by its homotopy class.

Acknowledgements: Thanks to Ludmil Katzarkov and Gang Tian for motivating questions and to Steve Kherchoff and Andras Stipsicz for valuable conversations. Paul Seidel kindly identified some errors in an earlier draft of the paper.

2 Shearing the hyperbolic disc

Our approach to (1.1) will be to analyse the action of the monodromy determining a Lefschetz fibration on the universal cover of a fixed smooth fibre². From elementary covering space theory we know that any homeomorphism of a topological space lifts to a homeomorphism of the universal cover. The lift is unique if we make the covering map a map of pointed spaces by introducing base-points. For a homeomorphism ϕ of a space X we shall write $\tilde{\phi}$ for any choice of lift to the universal cover. The uniformisation theorem allows us to view the two-dimensional fibre as a quotient of the hyperbolic disc D with its Poincaré metric by a discrete subgroup of the isometry group of D . Crucial to us will be the circle at infinity \mathbb{S}_∞^1 which naturally compactifies D . Most of the material of the following section is standard and can be found in work of Nielsen [11]; see also the accounts of Thurston's work in [3] and [16]. For completeness we review the necessary ideas. Fix a covering projection $D \xrightarrow{\pi} \Sigma_g$ and an identification of $\pi_1(\Sigma_g) \cong G$ with a discrete subgroup $G < \text{Isom}(D)$. Write e for the identity element of G .

The Poincaré metric on the unit disc is entirely specified by declaring that its isometry group consists of the Möbius transformations of the complex plane which preserve D . With respect to this metric, geodesics in D are given by arcs of circles which are orthogonal to the boundary \mathbb{S}_∞^1 . A pair of distinct points on \mathbb{S}_∞^1 determines a unique geodesic in D and hence geodesic $\gamma : (-\infty, \infty) \rightarrow \Sigma_g$ which may or may not close. Any choice of lift of a simple closed curve in Σ_g gives a curve in D meeting the boundary at two points which is bounded isotopic to the geodesic corresponding to this boundary pair. Since the group G acts on D preserving geodesics, it follows that the action extends to an action on \mathbb{S}_∞^1 by orientation-preserving homeomorphisms. For any $u \in G \setminus e$, the induced map $u_* : \mathbb{S}_\infty^1 \rightarrow \mathbb{S}_\infty^1$ has two fixed points, one attracting and one repelling; under iterations, all points on the circle save for the repelling fixed point are compressed near the attracting fixed point. (Such a homeomorphism of the disc and/or circle is called a *hyperbolic* map.) The geodesic determined by the two fixed points projects to a closed curve in Σ_g which is the unique geodesic in the free homotopy class associated to $u \in G$. As we vary over $u \in G$ the fixed points vary over a dense set of \mathbb{S}_∞^1 . Note importantly that the hyperbolic covering transformation has *no fixed points* in the interior of the disc D . The following is our principal tool.

(2.1) PROPOSITION:

- Let $\tau : \Sigma_g \rightarrow \Sigma_g$ be any homeomorphism of the surface. Then each lift of τ to D induces an orientation-preserving homeomorphism of \mathbb{S}_∞^1 , and

²Recall that we always assume that the fibre genus is at least two.

the set of homeomorphisms one obtains in this way depends only on the isotopy class of τ in the mapping class group.

- *The lifts of a Dehn twist about $C \subset \Sigma_g$ to D fall into two families: those which fix some point inside the interior of D , which then fix a component of the complement in D of a neighbourhood of the locus Λ_C of all lifts of C , and those which fix no point of $\text{Int}(D)$. The members of the first family are permuted by the conjugation action of G on the set of lifts.*
- *Let $\delta : \Sigma_g \rightarrow \Sigma_g$ be a positive Dehn twist about some curve $C \subset \Sigma_g$. Then any lift $\tilde{\delta}$ of δ to D which has a fixed point in the interior of D fixes a countable number of points of \mathbb{S}_∞^1 and moves all other points in the same sense (clockwise in the convention in which holomorphic Lefschetz fibrations have positive Dehn twist monodromies).*

PROOF: The homeomorphism induces an automorphism of G via $g \mapsto \tau g \tau^{-1}$ (which still covers the identity). Using this we can induce an automorphism of the end-points of the geodesics corresponding to elements of G ; since these form a dense subset of the circle we therefore have a homeomorphism of \mathbb{S}_∞^1 . It is easy to check that this preserves the cyclic ordering of the circle and is orientation-preserving. If we change τ by isotopy we will not change the isotopy class of the image of any curve under τ ; since there is a unique geodesic in each isotopy class of simple closed curve, and it is the action on these geodesics which determines the action on \mathbb{S}_∞^1 , the independence of the extension under isotopies follows.

We give an explicit description of the lifts which fix a region in the proof of the third part of the lemma; it is clear that these are permuted by the conjugation action of G . Now suppose a lift $\tilde{\delta}$ fixes some point of $\text{Int}(D)$. This lies inside some region of $D \setminus \Lambda_C$ - it cannot lie inside Λ_C since the Dehn twist itself does not preserve any point of the curve C (on which it acts by the antipodal map). For this particular region there is a unique lift $\tilde{\delta}_0$ of the twist which fixes the open set given by the intersection of the region and the exterior of a neighbourhood of the boundary lifts of C . The product $(\tilde{\delta})^{-1} \tilde{\delta}_0$ now covers the identity on Σ_g and is hence a deck transformation; but it also fixes a point of the interior of D , by construction, which means that it must be the trivial deck transformation $e \in G$. It follows that a lift of the Dehn twist fixing some interior point necessarily coincides with the lift fixing the region containing the point.

The homeomorphism τ_C given by twisting about C is supported inside an annular neighbourhood A of C ; standard covering space theory implies that it can be lifted to be the identity in a single component K of the universal cover of $\Sigma \setminus A$. Fix a base-point P inside the region K . Given the definition of the action of a mapping class on the circle at infinity, we may understand the effect of the Dehn twist about a curve C as follows. Choose a point p on \mathbb{S}_∞^1 ; visualise p as lying at twelve o'clock. This point, together with the base-point P , determines a quasigeodesic $L : [0, \infty) \rightarrow D$ and a quasigeodesic l in the surface Σ_g which may intersect C , perhaps infinitely often. The effect of the Dehn twist $l \mapsto \tau_C(l)$ is to insert a copy of C at each of these intersection points. Start at the image of P in Σ_g and consider moving along the new curve $\tau_C(l)$; then upstairs we start at P and move along L until we meet some lift of C , then we turn right and move along this lift of C until we meet the adjacent lift of l , then we turn left and move in the initial direction along this new parallel lift of l until we meet the next intersection point with a lift of C , turn right and so forth (draw a picture!). This process eventually meanders into \mathbb{S}_∞^1 , by definition giving rise to the image of p under the homeomorphism of \mathbb{S}_∞^1 induced by this particular lift

of the Dehn twist about C . Suppose for contradiction this image point $\tilde{\tau}_C(p)$ lies to the left of p (as viewed from inside the disc, i.e. anticlockwise along the circle from p). Then after some finite number of intersections with C and zig-zags in the above procedure, the extension of the lift of l on which we are travelling also has end-point anticlockwise from p . Suppose for simplicity that this happens in fact after just one zig-zag, that is that C and l meet only once in Σ_g . Then we have a geodesic triangle inside D comprising the arc of C and the two arcs of lifts of l , which by our assumptions must meet at some third point. But now two of the angles of our triangle (at the intersection points of the arc of C and of lifts of l) sum to π ; this is impossible in hyperbolic trigonometry.

For a larger number of zig-zags one can complete by induction, the crucial point being that the intersection angles of lifts of C and of l are always equal in the disc. It follows that all points that are moved at all are indeed moved in one particular sense, determined by the positivity of the Dehn twist homeomorphism downstairs. Moreover it follows that some points of the circle are moved by the lift $\tilde{\tau}_C$, coming from closed geodesics which do intersect C .

Finally, to see that countably many points are fixed, consider again the fixed lift K in D of the complement $\Sigma \setminus A$. The boundary of this complement downstairs lifts to give pairs of curves with equal endpoints which are endpoints of lifts of the geodesic in the free homotopy class of C . There are countably many of these in the boundary of K (of course the endpoints of all the lifts of C give a dense subset of \mathbb{S}_∞^1 but this is not true if we restrict to those at the boundary of the one region). Now the homeomorphism τ_C given by twisting about C is the identity inside K and hence preserves (setwise!) each of the boundary curves of K . It follows that the induced map on \mathbb{S}_∞^1 has a countable set of fixed points. ■

(2.2) REMARK: Note that it is *not* true that every lift of a Dehn twist has this nice behaviour at the circle at infinity. For if we compose a lift with an interior fixed point with some large power of a fixed hyperbolic deck transformation, the compression of \mathbb{S}_∞^1 near the attracting fixed point of the hyperbolic will destroy the clockwise shear. (This example was pointed out to me by Paul Seidel.) Fortunately the applications we have in mind necessitate choosing lifts of Dehn twists which *do* have interior fixed points.

So far the non-uniqueness of lifts presents a problem for applications. If we suppose that our fibration has a section, however, this problem is entirely removed. Recall that if we fix a base-point \tilde{p} in D covering a base-point p in Σ_g any homeomorphism τ of the base has a distinguished lift $\tilde{\tau}_{(p)}$ to D . If the homeomorphism fixes p then the lift fixes the distinguished point \tilde{p} (but in general induces a non-trivial permutation of the other lifts of p). A section of a Lefschetz fibration precisely amounts to a point $p \in \Sigma_g$ fixed by each of the Dehn twist monodromies δ_i . In fact more is true; the point is then fixed by the isotopy from the product homeomorphism $\prod \delta_i$ to the identity. If we lift each of the Dehn twists in turn, we induce a sequence of homeomorphisms of the circle whose product must be the identity. For the final isotopy does not have any effect on \mathbb{S}_∞^1 by the last part of (2.1), and so the product of the lifts $\prod \tilde{\delta}_i$ acting on \mathbb{S}_∞^1 must be the same as the lift of the identity, which is the result of that isotopy downstairs in Σ_g . But the whole process has fixed the point \tilde{p} and only the identity deck transformation fixes a point of D . Thus the final homeomorphism on \mathbb{S}_∞^1 co-incides with the extension to \mathbb{S}_∞^1 of the lift of the identity of Σ_g , which is clearly trivial. However, by lifting the various Dehn

twist homeomorphisms in turn we have collected another piece of information; a winding number, telling us how often we have spun the circle. (This winding number can be viewed as a translation number if we lift now to the universal cover \mathbb{R} of the circle \mathbb{S}_∞^1 but the meaning should be clear without this additional layer of notation.)

(2.3) LEMMA: *The sequence of homeomorphisms $\tilde{\delta}_i$ rotates the circle clockwise by $2\pi k$, where the positive integer $k = -s \cdot s$ is given by the negative of the self-intersection of the section s of the Lefschetz fibration defined by the point p .*

PROOF: Fix a small disc centred on \tilde{p} . Since \tilde{p} is fixed by all the homeomorphisms of the disc $\tilde{\delta}_i$ and the final isotopy, this disc is mapped eventually to another topological disc enclosing \tilde{p} . It follows that the boundary of a small disc around \tilde{p} is homotopy rotated a certain number of times by the product of the homeomorphisms; if we now take two points \tilde{p}, \tilde{q} which are nearby lifts of close points p, q in Σ_g then under the sequence of homeomorphisms the point \tilde{q} will move around \tilde{p} a number m times. This number is precisely the number of intersection points of the sections defined by p, q respectively, which is just the self-intersection $|s \cdot s|$. On the other hand, by considering a radial projection along geodesic arcs, since the homeomorphisms of the disc preserve the cyclic ordering on \mathbb{S}_∞^1 we see that this number is the same as the rotation number at the boundary of the disc. In more formal language, there is a short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \Gamma_{g,1} \rightarrow \Gamma_g^1 \rightarrow 0$$

where the middle term is the mapping class group for a surface with one boundary circle and the right hand term the mapping class group for a punctured surface. The copy of \mathbb{Z} is generated by the Dehn twist about the puncture; our Dehn twist monodromies naturally lift from Γ_g^1 to $\Gamma_{g,1}$ but their product may differ from the identity in $\Gamma_{g,1}$ by some integer in the kernel of the natural map $\Gamma_{g,1} \rightarrow \Gamma_g^1$, and this is the self-intersection of our section. There is another short exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \text{Homeo}^b(\mathbb{R}) \rightarrow \text{Homeo}(\mathbb{S}_\infty^1) \rightarrow 1$$

where all homeomorphisms are orientation-preserving and the middle group denotes the homeomorphisms of the real line f for which $f(x) + 2\pi = f(x + 2\pi)$. The copy of \mathbb{Z} is generated by translations. Then the statement of the lemma amounts to saying that the first short exact sequence is induced from the second under the representation $\Gamma_g^1 \rightarrow \text{Homeo}(\mathbb{S}^1)$.

Being explicit again, note from the description with p, q that the sign of the self-intersection of any section is determined by the sense of the rotations on \mathbb{S}_∞^1 . It follows that the sign of the self-intersection of the section is always the same for any Lefschetz fibration determined by positive Dehn twists; but we know of many fibrations containing sections of negative square, from blowing up base-points. ■

3 Irreducible Lefschetz fibrations

Since one can always introduce critical fibres in Lefschetz fibrations by blowing up points, it is standard to assume that all Lefschetz fibrations have no spherical components in fibres. The genus zero fibrations are then Hirzebruch surfaces

whilst the genus one fibrations are elliptic surfaces [1]. All the fibrations are holomorphic, and moreover all are given by fibre summing certain “irreducible” building blocks by identity diffeomorphisms. This prompted the following question:

(3.1) QUESTION: [Gang Tian] Is every simply-connected Lefschetz fibration a fibre sum of holomorphic Lefschetz fibrations?

From the known restrictions on the minimal length of mapping class group words, one can find examples showing that the assumption of simple connectivity is necessary [13]. Our next example shows that it is not sufficient:

(3.2) PROPOSITION: *Any pencil of curves of genus $g > 5$ on a symplectic non-Kähler K3 surface gives rise to a Lefschetz fibration which is not a Kähler sum.*

There are many - gauge-theory based - constructions of such fake K3 surfaces [5]; the Lefschetz pencils, and hence fibrations on blowing up base-points, are provided by Donaldson’s existence result. The fibres of the pencil may have arbitrarily large genus by scaling the symplectic form and computing with the adjunction formula.

PROOF: Let $Z' \rightarrow \mathbb{P}^1$ be the fibration constructed from our fake K3 surface. Note that $\sigma + e$ - the sum of signature and Euler characteristic - is invariant under blowing up and down. If we write, for contradiction, that $Z' = W_1 \#_{\text{Fibre}} W_2$ is a fibre sum (of non-trivial Lefschetz fibrations, that is fibrations with non-zero number of singular fibres), then by some easy computations we have

$$\sigma(Z') = \sigma(W_1) + \sigma(W_2); \quad e(Z') = e(W_1) + e(W_2) - 2e(F) \quad (3.3)$$

where F denotes a smooth fibre. It follows, in obvious notation, that

$$(\sigma_1 + e_1) + (\sigma_2 + e_2) = 8 + 2e(F) < 0 \quad (3.4)$$

and hence for W_1 say we have $\sigma_1 + e_1 < 0$. But this forces, by the classification of complex surfaces, the manifold W_1 to be the blow-up of an irrational ruled manifold $\Sigma_h \tilde{\times} \mathbb{S}^2$ for some h . Since $\pi_1(W_1)$ is $\pi_1(\Sigma_h)$ and also a quotient of $\pi_1(\Sigma_{g(F)})$ by vanishing cycles we see $h \leq g(F)$. Indeed if $h = g(F)$ then all the vanishing cycles for W_1 are nullhomotopic and it is a trivial piece in the decomposition, a contradiction. Hence $h < g(F)$.

We claim that in fact $2h \leq g(F)$. To see this, take a basis for the homology of the irrational ruled manifold comprising a section $[s]$ and fibre $[f]$ (the latter is canonically defined, the former only up to twisting $[s] \mapsto [s \pm nf]$). By the adjunction formula, note that any complex curve representing any section of the fibration $\Sigma_h \tilde{\times} \mathbb{S}^2 \rightarrow \Sigma_h$ has genus $g_{[s \pm nf]} = h$. Since $g(F) > h$ we must have $F = a[s] + b[f]$ with $a \neq 1$, and hence $a \geq 2$ since the symplectic form is positive on F . Finally, McDuff [10] has shown that for a symplectic form $\omega = P.D(a[s] + b[f])$ on an irrational ruled manifold W with fibre \mathbb{S} we have

$$\omega^2(W) > \omega(\mathbb{S})^2 \Rightarrow a^2[s]^2 + 2ab > a^2.$$

Applying adjunction to $F = a[s] + b[f]$ shows that

$$2g(F) - 2 = aK_W \cdot [s] - 2b + a^2[s]^2 + 2ab = a(2h - 2) + (a - 1)(a[s]^2 + 2b);$$

combining this with McDuff's result and $a \geq 2$ gives $g(F) \geq 2h$ as claimed.

Now $\sigma_1 + e_1$ (invariant under blow-ups) is equal to $2e(\Sigma_h)$ whilst $2e(\Sigma_h) - 2e(F) > 8$ from the original stipulation that $g(F) > 5$. It therefore follows from (3.4) that $\sigma_2 + e_2 < 0$ and hence that W_2 is also a blow-up of an irrational ruled manifold. From here the readers should be able to find their own conclusion to the proof; the combined restriction of having both pieces of a decomposition (blow-ups of) irrational ruled manifolds is too considerable. ■

The details of the proof are surprisingly technical. In fact there is a much more elementary obstruction at work. The Lefschetz fibrations produced by Donaldson's construction are *never* fibre summations. This was first proven by Stipsicz [15] using Seiberg-Witten theory, and in particular the fact that a symplectic manifold with $b_+ > 1$ cannot contain a symplectic sphere of positive square. The above example (historically the first) can be seen as a particular realisation of those kinds of method. In fact the general case follows easily from our remarks on the hyperbolic disc above. The proof will recover, as an addendum, the fact that symplectic spheres which arise as *sections* of Lefschetz fibrations can never have positive square.

(3.5) PROPOSITION:

- *No Lefschetz fibration can contain a section of positive square.*
- *If a Lefschetz fibration contains a section of square zero, it is a trivial product $\Sigma_g \times \mathbb{S}^2$.*
- *If there is a section of square -1 then the fibration cannot split as a non-trivial fibre sum.*

PROOF: The first two statements are immediate from the last remarks in the proof of (2.3). Suppose then for contradiction that the Lefschetz fibration $X \rightarrow \mathbb{P}^1$ contains a section of square (-1) and splits as a non-trivial fibre sum $X = W \#_{\text{Fibre}} W'$. Choose a positive relation

$$\delta_1 \dots \delta_r \gamma_1 \dots \gamma_s = 1$$

for X which is a factorised product of relations for the constituent pieces W, W' . Fix a covering projection $\pi : D \rightarrow \text{Fibre}$ as above and a point \tilde{p} in D which projects to the intersection of the section with the distinguished fibre. We have a canonical sequence of lifts of monodromies $\tilde{\delta}_i, \tilde{\gamma}_j$ whose product induces the single full rotation of \mathbb{S}_∞^1 by 2π . On the other hand, once we have lifted the first r monodromies, we have lifted a word which is the identity, and hence we must induce a hyperbolic covering transformation on \mathbb{S}_∞^1 by the arguments as above. Thus the sequence of lifts serves to factorise the 2π -rotation of \mathbb{S}_∞^1 as a product of two hyperbolic automorphisms, each of which is a factorised product of clockwise shears.

But this is impossible, from our description of these hyperbolic elements. The first hyperbolic element has two fixed points. Therefore under the factorisation into twists, these points have either made at least one rotation of the circle or they have been fixed throughout. In the former case, points on one side of the repelling fixed point must have moved by more than one full rotation of the circle, since the cyclic ordering is preserved after each twist. But then since the total translation is increasing, composing with the twists arising from W' cannot yield a total translation of one 2π rotation. So suppose that the two

fixed points have been fixed throughout all the Dehn twist lifts for W . They must then be moved by one rotation by the lifts for W' . However, by the same argument, the two fixed points for W' are now fixed by all these latter rotations, and so the fixed points for W cannot pass them in order to move around the boundary of the disc. ■

We should make one point clear: it is not the case that all sections of a given fibration must have the same square. Kähler genus two fibrations with no reducible fibres were classified by Chakiris (cf. [14]). From that classification, it follows that the fibre sum (always by the identity diffeomorphism) of four copies of the fibration associated to the genus two pencil on $K3$ is isomorphic to the fibre sum of six copies of the fibration on the rational manifold $(\mathbb{S}^2 \times \mathbb{S}^2) \# 12\overline{\mathbb{P}}^2$; they each have full monodromy group and 120 singular fibres. On the other hand, in these decompositions the holomorphic fibration is shown off with sections of square -4 and -6 respectively. There is no analogue of this phenomenon for elliptic fibrations, where the adjunction formula fixes the square of any section.

4 Invariant homology is not homotopic

The most classical way to detect that a symplectic manifold is not Kähler is to show that the first Betti number is odd. Recall that the first homology of a Lefschetz fibration comprises the homology classes in the fibre which are invariant under the monodromy action: the representation of $\pi_1(\mathbb{S}^2 \setminus \{\text{Crit}\})$ on $H_1(\text{Fibre})$ has a trivial subrepresentation $H_1(X)$. It has become clear that treating the vanishing cycles up to isotopy and not just homology gives access to additional information on the underlying manifold. This was one motivation for the following

(4.1) QUESTION: [Ludmil Katzarkov] Can there be a simple closed curve in a fibre of a Lefschetz fibration which is invariant (up to isotopy) under all the monodromy diffeomorphisms of the fibration?

Katzarkov *et al* [8] gave a negative answer, based on Hodge theory, for Kähler fibrations arising from pencils of high degree on surfaces with vanishing first Betti number. The methods of the previous section make short work of the more general case.

(4.2) PROPOSITION: *There is no finite set of simple closed curves with embedded union which is preserved to isotopy by the monodromy of any Lefschetz fibration.*

PROOF: We treat the case of a single curve first. Suppose $C \subset \Sigma_g$ is such a curve. Fix a lift of C to the disc extended to the boundary \mathbb{S}_∞^1 giving two distinguished points $a, b \in \mathbb{S}_\infty^1$. Since all the monodromy diffeomorphisms fix C and in particular all the δ_i fix C , we can choose lifts of each δ_i in turn which fix *the whole lift* \tilde{C} . To see this, note that we can always lift a homeomorphism to be the identity in one lift of the complement of its support, and here we choose all these lifts to contain \tilde{C} . It follows that under the various monodromies upstairs, a, b are always fixed. However, the product of these monodromies must be a deck transformation on \mathbb{S}_∞^1 . This is impossible; the two fixed points a, b must be the fixed points of the hyperbolic, but because all the Dehn twists have moved points in a single sense, we see that in small arcs either side of say a the points of \mathbb{S}_∞^1 are all moved clockwise. This is impossible if a is either an attracting or

repelling fixed point.

To see that we cannot permute a collection of simple closed curves $\gamma_1, \dots, \gamma_q$ we use induction, with the above being the base step. Fix a distinguished curve γ_1 ; by the above there must be some vanishing cycle ψ which has non-trivial geometric intersection number with γ_1 . Consider the curves $\gamma_1^{(n)}$ given by applying the n -th power of the Dehn twist about ψ to γ_1 . Eventually we must have some identity $\gamma_1^{(n)} = \gamma_1$ for some n . But then the group generated by the Dehn twists about γ_1, ψ has a relation: the twist about γ_1 and the n -th power of the twist about ψ commute. It is known however that the group generated by the Dehn twists about two curves with non-trivial geometric intersection number is free of rank two if the intersection number is greater than one and the braid group on three strings if the number is precisely one. No relation of the given form can exist in either of these cases. ■

We can put a different interpretation on this last result. For a simple closed curve to be invariant under the Dehn twists about each vanishing cycle it would have to be disjoint from all of these; conversely, anything disjoint would certainly be invariant.

(4.3) COROLLARY: *For any Lefschetz fibration, the vanishing cycles fill up the fibre; their complement is a bunch of discs.*

It follows that the pattern of vanishing cycles on the fibre Σ_g amounts (in a generic and transverse picture) to a certain four-valent graph on the surface which is the one-skeleton of a cell decomposition. One can show that one half the number of faces in this decomposition gives an upper bound on the number of distinct homotopy classes of section of the fibration, but in general the bounds one gets this way are not remotely sharp. More curious is the following observation. So far the particular hyperbolic structures on surfaces that we have considered have been irrelevant and certainly not canonical. However, Kherchoff showed in [9] that the geodesic length function is convex along earthquake paths. It followed that for any collection of simple closed curves which fills a surface, there is a distinguished element of Teichmüller space T_g on which the total length of the geodesic representatives of these curves is minimised. Counting repeated isotopy classes of vanishing cycle with multiplicity, this total length defines a value $l(V)$ for any collection of vanishing cycles defining a Lefschetz fibration. Such collections are indexed by a braid group Br_n as explained in [13].

(4.4) DEFINITION: *Let $f : X \rightarrow \mathbb{S}^2$ be a genus g Lefschetz fibration defined by a family of positive relations $I(\text{Br}_n)$ indexed by a suitable braid group. The length $l(X, f)$ is the minimum $\min\{l(V_i) : i \in I(\text{Br}_n)\}$ and is an invariant of the Lefschetz fibration.*

Such invariants are unlikely to have subtle geometric content, since they do not depend on the *order* of the Dehn twist monodromies, and only on the supporting curves. Nonetheless it would be interesting to understand precisely what if anything they do capture. For instance, the length may plausibly be related to the minimal possible value of the total number of geometric intersections of the vanishing cycles, as we vary over the braid group. This latter invariant is related to Floer theory, and the sum of the ranks of the Floer homologies over all pairs of distinct vanishing cycles in the fibre.

5 Finitely many sections

Manin's theorem [7] asserts that a non-isotrivial holomorphic fibration has only finitely many holomorphic sections. We have the following analogue for Lefschetz fibrations:

(5.1) THEOREM: *Let $f : X \rightarrow \mathbb{S}^2$ be a symplectic Lefschetz fibration. Then f admits only finitely many homotopy classes of geometric section.*

PROOF: Any section can be trivialised over a large disc in the base, containing the critical values, and viewed as a point p in a fixed fibre Σ which is invariant under all the monodromy maps. That is, p is disjoint from the support of all the Dehn twist monodromies. The product of these is isotopic to the identity; under the isotopy, which determines how to close the fibration over the disc to one over the sphere, p moves through some loop $u \in \pi_1(\Sigma)$. There is a section through p iff this loop is nullhomotopic in Σ .

It follows that if we perturb p inside a component D of $\Sigma \setminus \{C_i\}$, where the C_i are the vanishing cycles of the fibration transported to lie in the fixed fibre, then the loop u associated to p is unchanged. From this it is easy to see that if there is a section through p its homotopy class is independent of the choice of p in D . So the number of homotopy classes of section is bounded above by the number of cells in the cell decomposition of Σ defined by the vanishing cycles, and is in particular finite. ■

One can improve the actual bound on the number of homotopy classes of sections, but the results seem a long way from being sharp in examples. If one is given a component D for which there is a section of the fibration, it is in principle mechanical to determine which other regions of $\Sigma \setminus \{C_i\}$ admit sections. For taking a path between points in the two regions, and the image of the path under all the monodromy twists, we have a loop in Σ which should be trivial in $\pi_1(\Sigma)$. Such a relation amongst the vanishing cycles is equivalent to the existence of a suitable closed polygon in the hyperbolic disc. Note finally that the *existence* question for sections of Lefschetz fibrations remains open.

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